

ON THE DIRICHLET'S TYPE OF EULERIAN POLYNOMIALS

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ABSTRACT. In the present paper, we introduce Eulerian polynomials attached to χ by using p -adic q -integral on \mathbb{Z}_p . Also, we give new interesting identities via the generating functions of Dirichlet's type of Eulerian polynomials. After, by applying Mellin transformation to this generating function of Dirichlet's type of Eulerian polynomials, we derive L -function for Eulerian polynomials which interpolates of Dirichlet's type of Eulerian polynomials at negative integers.

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1. Introduction

Recently, Kim *et al.* have studied to Eulerian polynomials. They gave not only Witt's formula for Eulerian polynomials but also relations between Genocchi, Tangent and Euler numbers (for more details, see [1]). In arithmetic works of T. Kim have introduced many different generating functions of families of Bernoulli, Euler, Genocchi numbers and polynomials by using p -adic q -integral on \mathbb{Z}_p (see [1-5]). After, many mathematicians are motivated from his papers and introduced new generating function for special functions (for more information about this subject, see [16-28]). Y. Simsek also gave new q -twisted Euler numbers and polynomials and (h, q) -Bernoulli numbers and polynomials by using Kim's p -adic q -integral on \mathbb{Z}_p . He also derived some interesting properties in his works [26], [27], [28].

The p -adic q -integral on \mathbb{Z}_p was originally defined by Kim. He also investigated that p -adic q -integral on \mathbb{Z}_p is related to non-Archimedean combinatorial analysis in mathematical physics. That is, the functional equation of the q -zeta function, the q -Stirling numbers and q -Mahler theory and so on (for details, see [5], [6]).

We firstly list some properties of familiar Eulerian polynomials for sequel of this paper as follows:

As it is well-known, the Eulerian polynomials, $\mathcal{A}_n(x)$ are given by means of the following generating function:

$$(1) \quad e^{\mathcal{A}(x)t} = \sum_{n=0}^{\infty} \mathcal{A}_n(x) \frac{t^n}{n!} = \frac{1-x}{e^{t(1-x)} - x}$$

where $\mathcal{A}^n(x) := \mathcal{A}_n(x)$ as symbolic. To find Eulerian polynomials, it has the following recurrence relation:

$$(2) \quad (\mathcal{A}(t) + (t-1))^n - t\mathcal{A}_n(t) = \begin{cases} 1-t & \text{if } n=0 \\ 0 & \text{if } n \neq 0, \end{cases}$$

(for details, see [1]).

Suppose that p be a fixed odd prime number. Throughout this paper, we use the following notations. By \mathbb{Z}_p , we denote the ring of p -adic rational integers, \mathbb{Q} denotes the field of rational numbers, \mathbb{Q}_p denotes the field of p -adic rational numbers, and \mathbb{C}_p denotes the completion of algebraic closure of \mathbb{Q}_p . Let \mathbb{N} be the set of natural numbers and $\mathbb{N}^* = \mathbb{N} \cup \{0\}$.

The p -adic absolute value is defined by

$$|p|_p = \frac{1}{p}.$$

In this paper we assume $|q-1|_p < 1$ as an indeterminate. Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on \mathbb{Z}_p . For a positive integer d with $(d, p) = 1$, set

$$\begin{aligned} X &= X_m = \varprojlim_m \mathbb{Z}/dp^m\mathbb{Z}, \\ X^* &= \bigcup_{\substack{0 < a < dp \\ (a, p) = 1}} a + dp\mathbb{Z}_p \end{aligned}$$

and

$$a + dp^m\mathbb{Z}_p = \{x \in X \mid x \equiv a \pmod{dp^m}\},$$

where $a \in \mathbb{Z}$ satisfies the condition $0 \leq a < dp^m$.

Firstly, for introducing fermionic p -adic q -integral, we need some basic information which we state here. A measure on \mathbb{Z}_p with values in a p -adic Banach space B is a continuous linear map

$$f \mapsto \int f(x)\mu = \int_{\mathbb{Z}_p} f(x)\mu(x)$$

from $C^0(\mathbb{Z}_p, \mathbb{C}_p)$, (continuous function on \mathbb{Z}_p) to B . We know that the set of locally constant functions from \mathbb{Z}_p to \mathbb{Q}_p is dense in $C^0(\mathbb{Z}_p, \mathbb{C}_p)$ so.

Explicitly, for all $f \in C^0(\mathbb{Z}_p, \mathbb{C}_p)$, the locally constant functions

$$f_n = \sum_{i=0}^{p^m-1} f(i)1_{i+p^m\mathbb{Z}_p} \rightarrow f \text{ in } C^0$$

Now, set $\mu(i + p^m\mathbb{Z}_p) = \int_{\mathbb{Z}_p} 1_{i+p^m\mathbb{Z}_p}\mu$. Then $\int_{\mathbb{Z}_p} f\mu$, is given by the following Riemannian sum

$$\int_{\mathbb{Z}_p} f\mu = \lim_{m \rightarrow \infty} \sum_{i=0}^{p^m-1} f(i)\mu(i + p^m\mathbb{Z}_p)$$

The following q -Haar measure is defined by Kim in [3] and [5]:

$$\mu_q(a + p^m\mathbb{Z}_p) = \frac{q^a}{[p^m]_q}$$

So, for $f \in UD(\mathbb{Z}_p)$, the p -adic q -integral on \mathbb{Z}_p is defined by Kim as follows:

$$(3) \quad I_q(f) = \int_{\mathbb{Z}_p} f(\eta) d\mu_q(\eta) = \lim_{n \rightarrow \infty} \frac{1}{[p^n]_q} \sum_{\eta=0}^{p^n-1} q^\eta f(\eta).$$

The bosonic integral is considered as the bosonic limit $q \rightarrow 1$, $I_1(f) = \lim_{q \rightarrow 1} I_q(f)$. In [10], [11] and [12], similarly, the p -adic fermionic integration on \mathbb{Z}_p defined by Kim as follows:

$$(4) \quad I_{-q}(f) = \lim_{q \rightarrow -q} I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x).$$

By (4), we have the following well-known integral equation:

$$(5) \quad q^n I_{-q}(f_n) + (-1)^{n-1} I_{-q}(f) = [2]_q \sum_{l=0}^{n-1} (-1)^{n-1-l} q^l f(l)$$

Here $f_n(x) := f(x+n)$. By (5), we have the following equalities:
If n odd, then

$$(6) \quad q^n I_{-q}(f_n) + I_{-q}(f) = [2]_q \sum_{l=0}^{n-1} (-1)^l q^l f(l).$$

If n even, then we have

$$(7) \quad I_{-q}(f) - q^n I_{-q}(f_n) = [2]_q \sum_{l=0}^{n-1} (-1)^l q^l f(l).$$

Substituting $n = 1$ into (6), we readily see the following

$$(8) \quad q I_{-q}(f_1) + I_{-q}(f) = [2]_q f(0).$$

Replacing q by q^{-1} in (8), we easily derive the following

$$(9) \quad I_{-q^{-1}}(f_1) + q I_{-q^{-1}}(f) = [2]_q f(0).$$

In [1], Kim *et al.* is considered $f(x) = e^{-x(1+q)t}$ in (9), then they gave Witt's formula of Eulerian polynomials as follows:

For $n \in \mathbb{N}^*$,

$$(10) \quad I_{-q^{-1}}(x^n) = \frac{(-1)^n}{(1+q)^n} \mathcal{A}_n(-q).$$

Now also, we consider $I_{-q^{-1}}(\chi(x)x^n)$ in the next section. We shall call as Dirichlet's type of Eulerian polynomials. After we shall give arithmetic properties for Dirichlet's type of Eulerian polynomials.

2. On the Dirichlet's type of Eulerian polynomials

Firstly, we consider the following equality by using (6): For d odd natural numbers,

$$(11) \quad \int_{\mathbb{Z}_p} f(x+d) d\mu_{-q^{-1}}(x) + q^d \int_{\mathbb{Z}_p} f(x) d\mu_{-q^{-1}}(x) \\ = [2]_q \sum_{0 \leq l \leq d-1} (-1)^l q^{d-l+1} f(l).$$

Let χ be a Dirichlet's character of conductor d , which is any multiple of p ($=\text{odd}$). Then, substituting $f(x) = \chi(x) e^{-x(1+q)t}$ in (11), then we compute as follows:

$$\begin{aligned} \int_{\mathbb{Z}_p} \chi(x+d) e^{-(x+d)(1+q)t} d\mu_{-q^{-1}}(x) + q^d \int_{\mathbb{Z}_p} \chi(x) e^{-x(1+q)t} d\mu_{-q^{-1}}(x) \\ = [2]_q \sum_{0 \leq l \leq d-1} (-1)^l q^{d-l+1} \chi(l) e^{-l(1+q)t} \end{aligned}$$

After some applications, we discover the following

$$(12) \quad \int_{\mathbb{Z}_p} \chi(x) e^{-x(1+q)t} d\mu_{-q^{-1}}(x) = [2]_q \sum_{l=0}^{d-1} (-1)^l q^{d-l+1} \chi(l) \frac{e^{-l(1+q)t}}{e^{-d(1+q)t} + q^d}.$$

Let $\mathcal{F}_q(t | \chi) = \sum_{n=0}^{\infty} \mathcal{A}_{n,\chi}(-q) \frac{t^n}{n!}$. Then, we introduce the following definition of generating function of Dirichlet's type of Eulerian polynomials.

Definition 1. For $n \in \mathbb{N}^*$, then we define the following:

$$(13) \quad \sum_{n=0}^{\infty} \mathcal{A}_{n,\chi}(-q) \frac{t^n}{n!} = [2]_q \sum_{l=0}^{d-1} (-1)^l q^{d-l+1} \chi(l) \frac{e^{-l(1+q)t}}{e^{-d(1+q)t} + q^d}.$$

By (12) and (13), we state the following theorem which is the Witt's formula for Dirichlet's type of Eulerian polynomials.

Theorem 2.1. The following identity holds true:

$$(14) \quad I_{-q^{-1}}(\chi(x) x^n) = \frac{(-1)^n}{(1+q)^n} \mathcal{A}_{n,\chi}(-q).$$

By using (13), we discover the following applications:

$$\begin{aligned} \sum_{n=0}^{\infty} \mathcal{A}_{n,\chi}(-q) \frac{t^n}{n!} &= [2]_q \sum_{l=0}^{d-1} (-1)^l q^{d-l+1} \chi(l) \frac{e^{-l(1+q)t}}{e^{-d(1+q)t} + q^d} \\ &= [2]_q \sum_{l=0}^{d-1} (-1)^l q^{-l+1} \chi(l) e^{-l(1+q)t} \sum_{m=0}^{\infty} (-1)^m q^{-md} e^{-md(1+q)t} \\ &= q [2]_q \sum_{m=0}^{\infty} \sum_{l=0}^{d-1} (-1)^{l+md} \chi(l+md) q^{-(l+md)} e^{-(l+md)(1+q)t} \\ &= q [2]_q \sum_{m=0}^{\infty} (-1)^m \chi(m) q^{-m} e^{-m(1+q)t}. \end{aligned}$$

Thus, we get the following theorem.

Theorem 2.2. The following

$$(15) \quad \mathcal{F}_q(t | \chi) = \sum_{n=0}^{\infty} \mathcal{A}_{n,\chi}(-q) \frac{t^n}{n!} = [2]_q \sum_{m=0}^{\infty} \frac{(-1)^m \chi(m) q^{-m} e^{-m(1+q)t}}{q^{m-1}}$$

is true.

By considering Taylor expansion of $e^{-m(1+q)t}$ in (15), we procure the following theorem.

Theorem 2.3. For $n \in \mathbb{N}$, then we have

$$(16) \quad \frac{(-1)^n}{q(1+q)^{n+1}} \mathcal{A}_{n,\chi}(-q) = \sum_{m=1}^{\infty} \frac{(-1)^m \chi(m) m^n}{q^m}.$$

From (14) and (16), we easily derive the following corollary:

Corollary 2.4. For $n \in \mathbb{N}$, then we procure the following

$$\lim_{m \rightarrow \infty} \sum_{x=1}^{p^m-1} \frac{(-1)^x \chi(x) x^n}{q^x} = 2 \sum_{m=1}^{\infty} \frac{(-1)^m \chi(m) m^n}{q^{m-2}}.$$

Now, we give distribution formula for Dirichlet's type of Eulerian polynomials by using p -adic q -integral on \mathbb{Z}_p , as follows:

$$\begin{aligned} \int_{\mathbb{Z}_p} \chi(x) x^n d\mu_{-q^{-1}}(x) &= \lim_{m \rightarrow \infty} \frac{1}{[dp^m]_{-q^{-1}}} \sum_{x=0}^{dp^m-1} (-1)^x \chi(x) x^n q^{-x} \\ &= \frac{d^n}{[d]_{-q^{-1}}} \sum_{a=0}^{d-1} (-1)^a \chi(a) q^{-a} \left(\lim_{m \rightarrow \infty} \frac{1}{[p^m]_{-q^{-d}}} \sum_{x=0}^{p^m-1} (-1)^x \left(\frac{a}{d} + x\right)^n q^{-dx} \right) \\ &= \frac{d^n}{[d]_{-q^{-1}}} \sum_{a=0}^{d-1} (-1)^a \chi(a) q^{-a} \int_{\mathbb{Z}_p} \left(\frac{a}{d} + x\right)^n d\mu_{-q^{-d}}(x). \end{aligned}$$

Thus, we state the following theorem.

Theorem 2.5. The following identity holds true:

$$(17) \quad \frac{(-1)^n}{(1+q)^n} \mathcal{A}_{n,\chi}(-q) = \frac{d^n}{[d]_{-q^{-1}}} \sum_{a=0}^{d-1} (-1)^a \chi(a) q^{-a} \int_{\mathbb{Z}_p} \left(\frac{a}{d} + x\right)^n d\mu_{-q^{-d}}(x).$$

From this, we notice that the above equation is related to q -Genocchi polynomials with weight zero, $\tilde{G}_{n,q}(x)$, and q -Euler polynomials with weight zero, $\tilde{E}_{n,q}(x)$, which is defined by Araci *et al.* and Kim and Choi in [25] and [15] as follows:

$$(18) \quad \frac{\tilde{G}_{n+1,q}(x)}{n+1} = \lim_{m \rightarrow \infty} \frac{1}{[p^m]_{-q}} \sum_{y=0}^{p^m-1} (-1)^y (x+y)^n q^y$$

and

$$(19) \quad \tilde{E}_{n,q}(x) = \int_{\mathbb{Z}_p} (x+y)^n d\mu_{-q}(y).$$

By expressions of (17), (18) and (19), we easily discover the following corollary.

Corollary 2.6. For $n \in \mathbb{N}^*$, then we have

$$\frac{(-1)^n}{(1+q)^n} \mathcal{A}_{n,\chi}(-q) = \frac{d^n}{(n+1)[d]_{-q^{-1}}} \sum_{a=0}^{d-1} (-1)^a \chi(a) q^{-a} \tilde{G}_{n+1,q^{-d}}\left(\frac{a}{d}\right)$$

Moreover,

$$\frac{(-1)^n}{(1+q)^n} \mathcal{A}_{n,\chi}(-q) = \frac{d^n}{[d]_{-q^{-1}}} \sum_{a=0}^{d-1} (-1)^a \chi(a) q^{-a} \tilde{E}_{n,q^{-d}}\left(\frac{a}{d}\right).$$

3. On the Eulerian- L function

Our goal in this section is to introduce Eulerian- L function by applying Mellin transformation to the generating function of Dirichlet's type of Eulerian polynomials. By (15), for $s \in \mathbb{C}$, we define the following

$$L_E(s | \chi) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \mathcal{F}_q(t | \chi) dt$$

where $\Gamma(s)$ is the Euler Gamma function. It becomes as follows:

$$\begin{aligned} L_E(s | \chi) &= q[2]_q \sum_{m=0}^{\infty} (-1)^m \chi(m) q^{-m} \left\{ \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-m(1+q)t} dt \right\} \\ &= \frac{q}{(1+q)^{s-1}} \sum_{m=1}^{\infty} \frac{(-1)^m \chi(m)}{q^m m^s} \end{aligned}$$

So, we give definition of Eulerian L -function as follows:

Definition 2. For $s \in \mathbb{C}$, then we have

$$(20) \quad L_E(s | \chi) = \frac{q}{(1+q)^{s-1}} \sum_{m=1}^{\infty} \frac{(-1)^m \chi(m)}{q^m m^s}.$$

Substituting $s = -n$ into (16), then, relation between Eulerian L -function and Dirichlet's type of Eulerian polynomials are given by the following theorem.

Theorem 3.1. The following equality holds true:

$$L_E(-n | \chi) = \begin{cases} -\mathcal{A}_{n,\chi}(-q) & \text{if } n \text{ odd,} \\ \mathcal{A}_{n,\chi}(-q) & \text{if } n \text{ even.} \end{cases}$$

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